

**THE CANTOR GAME: WINNING
STRATEGIES AND DETERMINACY**

by

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0. ABSTRACT

In [1] Grossman and Turett define the Cantor game. In [2] Matt Baker proves several results about the Cantor game and poses three challenging questions about it:

Do there exist uncountable subsets of $[0, 1]$ for which:

- (1) Alice does not have a winning strategy;*
- (2) Bob has a winning strategy;*
- (3) neither Alice nor Bob has a winning strategy?*

In this paper we show that the answers to these questions depend upon which axioms of set theory are assumed. Specifically, if we assume the Axiom of Determinacy in addition to the Zermelo-Fraenkel axioms, then the answer to all three questions is “no.” If instead we assume the Zermelo-Fraenkel axioms together with the Axiom of Choice, then the answer to questions 1 and 3 is “yes,” and the answer to question 2 is likely to be “no.”

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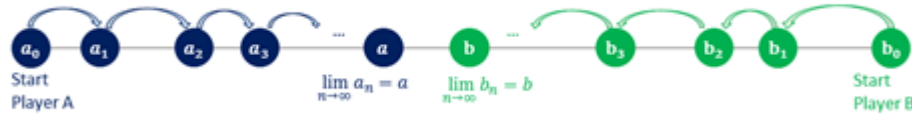


FIGURE 1. A play of the Cantor game. Player A wins if $a \in S$.

1. INTRODUCTION

In their paper [1] Grossman and Turett define the Cantor game. In [2] Matt Baker proves several results about the Cantor game and poses several challenging questions about it. The Cantor game is an infinite game played on the real line by two players, A (Alice) and B (Bob). Let a_0 and b_0 be fixed real numbers such that $a_0 < b_0$, and in the interval $[a_0, b_0]$ fix a subset S , which we call the *target set*. (In the original version of the Cantor game, a_0 and b_0 are chosen to be 0 and 1, respectively. In this case we take them to be arbitrary real numbers.) Initially, player A chooses a_0 , and then player B chooses b_0 . On the n th turn, for $n \geq 1$, player A chooses a_n with the property that $a_{n-1} < a_n < b_{n-1}$, and then player B chooses b_n such that $a_n < b_n < b_{n-1}$. Since the sequence $(a_n)_{n \geq 0}$ is strictly increasing and bounded above by b_0 , $\lim_{n \rightarrow \infty} a_n = a$ exists, and since $(b_n)_{n \geq 0}$ is strictly decreasing and bounded below by a_0 , $\lim_{n \rightarrow \infty} b_n = b$ exists. Since $a_n < b_n$ for all n , we also have $a \leq b$. In the end, if $a \in S$, player A wins; otherwise, if $a \notin S$, player B wins. Figure 1 illustrates a play of the Cantor game.

Baker uses the Cantor game to prove that the closed interval $[0, 1]$ is uncountable [2, p.377] and that every perfect set is uncountable [2, p.378]. He also shows that player B has a winning strategy when the target set S is countable [2, p.377]. After proving these results he poses three challenging questions [2, p.379]:

Do there exist uncountable subsets of $[0, 1]$ for which:

- (1) *Alice does not have a winning strategy;*
- (2) *Bob has a winning strategy;*
- (3) *neither Alice nor Bob has a winning strategy?*

We will show that the answers to these questions depend upon which axioms of set theory we assume. First we assume only the Zermelo-Fraenkel axioms of set theory (ZF) and prove some results about winning strategies for the Cantor game. Then assuming the Axiom of Determinacy in addition to ZF (ZF + AD), we shall prove that the answer to all three questions is “no.” Finally, assuming the Axiom of Choice in addition to ZF (ZFC), we will show that the answer to questions 1 and 3 is “yes,” and we will give evidence that the answer to question 2 is likely to be “no.”

2. STRATEGIC DEFINITIONS

Thomas Jech begins his monograph on axiomatic set theory by discussing the eight Zermelo-Fraenkel (ZF) axioms. He then develops some consequences of those axioms and shows how the real numbers can be constructed without additional assumptions [4]. This allows us to see which standard results of elementary real analysis follow from the ZF axioms. In this section and the following two sections we assume only the eight ZF axioms. Once we have proved several fascinating results about the Cantor game, we shall assume additional axioms and provide answers to our three questions.

Because our answers will require precise definitions and properties of strategies and generalized Cantor sets, we begin by defining these fundamental concepts. Our first definition was inspired by Oxtoby's work on the related Banach-Mazur game in [3, p.27]. In the following definitions we consider the Cantor game on $[a_0, b_0]$, where a_0 and b_0 are fixed real numbers satisfying $a_0 < b_0$, with an uncountable target set S .

Definition 1. A strategy for player A is an infinite sequence of real-valued functions $(f_n)_{n \geq 0}$. The function f_0 is required to have the domain $\{(a_0, b_0)\}$, and its value must satisfy $a_0 < f_0(a_0, b_0) < b_0$. For each $n \geq 2$ the function f_{n-1} has domain $\{(a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1}) \mid a_0 < a_1 < \dots < a_{n-1} < b_{n-1} < b_{n-2} < \dots < b_0\}$. If $a_n = f_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1})$, then we require that $a_{n-1} < a_n < b_{n-1}$.

Likewise, a strategy for player B is a sequence of real-valued functions $(g_n)_{n \geq 0}$. The function g_0 is required to have the domain $\{(a_0, b_0, a_1) \mid a_0 < a_1 < b_0\}$, and its value must satisfy $a_1 < g_0(a_0, b_0, a_1) < b_0$. For each $n \geq 2$ the function g_{n-1} has domain $\{(a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n) \mid a_0 < a_1 < \dots < a_n < b_{n-1} < b_{n-2} < \dots < b_0\}$. Furthermore, if $b_n = g_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1}, a_n)$, then we require that $a_n < b_n < b_{n-1}$.

Definition 2. A play of the game is an ordered pair of sequences $((a_n)_{n \geq 0}, (b_n)_{n \geq 0})$, where $a_{n-1} < a_n < b_n < b_{n-1}$ for each $n \geq 1$. The play is consistent with a given strategy for player A $(f_n)_{n \geq 0}$ if and only if $a_n = f_{n-1}(a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1})$ for all $n \geq 1$. Similarly, the play is consistent with the strategy for player B $(g_n)_{n \geq 0}$ if and only if $b_n = g_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1}, a_n)$ for all $n \geq 1$.

Definition 3. The limit set of a strategy $(f_n)_{n \geq 0}$ for player A is defined as follows:

$$L((f_n)) = \{ \lim_{n \rightarrow \infty} a_n \mid ((a_n), (b_n)) \text{ is a play of the game consistent with } (f_n) \}.$$

Likewise, the limit set of a strategy $(g_n)_{n \geq 0}$ for player B is defined as follows:

$$L((g_n)) = \{ \lim_{n \rightarrow \infty} a_n \mid ((a_n), (b_n)) \text{ is a play of the game consistent with } (g_n) \}.$$

We define a strategy $(f_n)_{n \geq 0}$ for player A to be a winning strategy if and only if $L((f_n)) \subset S$. A strategy $(g_n)_{n \geq 0}$ for player B is said to be a winning strategy if and only if $L((g_n)) \subset [a_0, b_0] - S$.

3. GENERALIZED CANTOR SETS

We define the concept of a generalized Cantor set as follows.

Definition 4. Let $I = [c, d]$, where $c < d$, and suppose that $(e_n)_{n \geq 1}$ is a strictly decreasing sequence of positive real numbers that converges to 0. For each $n \geq 1$ and each of the 2^n finite sequences of 0s and 1s, (i_1, i_2, \dots, i_n) , let $I_{i_1, i_2, \dots, i_n} = [c_{i_1, i_2, \dots, i_n}, d_{i_1, i_2, \dots, i_n}]$ be a closed interval. Suppose that the following properties hold:

- (1) $0 < d_{i_1, \dots, i_n} - c_{i_1, \dots, i_n} < e_n$ for all $n \geq 1$ and all $i_1, \dots, i_n \in \{0, 1\}$;
- (2) $I_0, I_1 \subset I$ and $I_{i_1, \dots, i_n} \subset I_{i_1, \dots, i_{n-1}}$ for all $n \geq 2$ and all $i_1, \dots, i_n \in \{0, 1\}$;
- (3) $I_0 \cap I_1 = \emptyset$ and $I_{i_1, \dots, i_{n-1}, 0} \cap I_{i_1, \dots, i_{n-1}, 1} = \emptyset$ for all $n \geq 2$ and all $i_1, \dots, i_{n-1} \in \{0, 1\}$.

For each $n \geq 1$ let

$$C_n = \bigcup_{i_1, \dots, i_n \in \{0, 1\}} I_{i_1, i_2, \dots, i_n},$$

and define

$$C = \bigcap_{n \geq 1} C_n.$$

Then C is called the generalized Cantor set generated by the collection of intervals

$$\mathcal{C} = \{I_{i_1, \dots, i_n} \mid n \geq 1 \text{ and } i_1, \dots, i_n \in \{0, 1\}\}.$$

Generalized Cantor sets have many of the familiar properties of the Cantor middle-thirds set. Before stating our main results on generalized Cantor sets, we first prove a lemma.

Lemma 1. Let C be the generalized Cantor set generated by the collection of intervals \mathcal{C} in Definition 4. Then for any $n \geq 1$, if $I_{i_1, \dots, i_n} \cap I_{j_1, \dots, j_n} \neq \emptyset$, then $i_1 = j_1, \dots, i_n = j_n$.

Proof. When $n = 1$, if $i_1 \neq j_1$, then $I_{i_1} \cap I_{j_1} = \emptyset$, and so the result holds. Now assume that the result holds for $n - 1$. Suppose that $x \in I_{i_1, \dots, i_n} \cap I_{j_1, \dots, j_n}$. By (2) of Definition 4 we see that $x \in I_{i_1, \dots, i_{n-1}} \cap I_{j_1, \dots, j_{n-1}}$. Now, by the induction hypothesis, $i_1 = j_1, \dots, i_{n-1} = j_{n-1}$. Hence $x \in I_{i_1, \dots, i_{n-1}, i_n} \cap I_{i_1, \dots, i_{n-1}, j_n}$. By (3) of Definition 4, these intervals are disjoint if $i_n \neq j_n$. It follows that $i_n = j_n$, and so the result is true by induction. \square

Theorem 1. Let C be the generalized Cantor set generated by the collection of intervals \mathcal{C} in Definition 4. Then the following properties hold:

- (1) If $(i_n)_{n \geq 1}$ is any sequence in $\{0, 1\}$, then $\lim_{n \rightarrow \infty} c_{i_1, \dots, i_n} = \lim_{n \rightarrow \infty} d_{i_1, \dots, i_n}$;
- (2) $x \in C$ if and only if there is a unique sequence $(i_n)_{n \geq 1}$ in $\{0, 1\}$ such that $\{x\} = \bigcap_{n \geq 1} I_{i_1, \dots, i_n}$. Furthermore, since $c_{i_1, \dots, i_n} \leq x \leq d_{i_1, \dots, i_n}$ for all n and the two sequences have equal limits, we must have $\lim_{n \rightarrow \infty} c_{i_1, \dots, i_n} = x = \lim_{n \rightarrow \infty} d_{i_1, \dots, i_n}$;

(3) C is a perfect set.

Proof. (1) Let $(i_n)_{n \geq 1}$ be any sequence in $\{0, 1\}$. By (2) of Definition 4, $I_{i_1, \dots, i_n} \subset I_{i_1, \dots, i_{n-1}}$, so that $c_{i_1, \dots, i_{n-1}} \leq c_{i_1, \dots, i_n}$. Hence $(c_{i_1, \dots, i_n})_{n \geq 1}$ is a monotonically increasing sequence bounded above by d , which implies that $\lim_{n \rightarrow \infty} c_{i_1, \dots, i_n}$ exists. A similar argument shows that $\lim_{n \rightarrow \infty} d_{i_1, \dots, i_n}$ exists. Since $\lim_{n \rightarrow \infty} e_n = 0$ and $0 < d_{i_1, \dots, i_n} - c_{i_1, \dots, i_n} < e_n$ by (1) of Definition 4, we have $\lim_{n \rightarrow \infty} (d_{i_1, \dots, i_n} - c_{i_1, \dots, i_n}) = 0$, and so $\lim_{n \rightarrow \infty} c_{i_1, \dots, i_n} = \lim_{n \rightarrow \infty} d_{i_1, \dots, i_n}$.

(2) Suppose that $x \in C$; then $x \in C_n$ for all $n \geq 1$. We proceed by induction on n . When $n = 1$, we know that $x \in C_1 = I_0 \cup I_1$, and we choose $i_1 \in \{0, 1\}$ such that $x \in I_{i_1}$. Now assume that i_1, i_2, \dots, i_{n-1} have been chosen such that $x \in I_{i_1}, x \in I_{i_1, i_2}, \dots, x \in I_{i_1, \dots, i_{n-1}}$. Since $x \in C_n$, there exist $j_1, j_2, \dots, j_{n-1}, i_n \in \{0, 1\}$ such that $x \in I_{j_1, \dots, j_{n-1}, i_n}$. By (2) of Definition 4, $x \in I_{j_1, \dots, j_{n-1}}$. By Lemma 1 this implies that $i_1 = j_1, \dots, i_{n-1} = j_{n-1}$. Thus $x \in I_{i_1, \dots, i_n}$. By induction, it follows that we have defined a sequence $(i_n)_{n \geq 1}$ such that $x \in \bigcap_{n \geq 1} I_{i_1, \dots, i_n}$.

Next, assume that $y \in \bigcap_{n \geq 1} I_{i_1, \dots, i_n}$. Then $x, y \in [c_{i_1, \dots, i_n}, d_{i_1, \dots, i_n}]$ for $n \geq 1$, and so $|y - x| \leq d_{i_1, \dots, i_n} - c_{i_1, \dots, i_n} < e_n$ for all n . Since $(e_n)_{n \geq 1}$ converges to 0, it follows that $|y - x| = 0$, and so $y = x$. Thus there exists a sequence $(i_n)_{n \geq 1}$ such that $\{x\} = \bigcap_{n \geq 1} I_{i_1, \dots, i_n}$. Now assume that there is another sequence $(j_n)_{n \geq 1}$ having the same property, and let $n \geq 1$ be given. Then $x \in I_{i_1, \dots, i_n} \cap I_{j_1, \dots, j_n}$, and so by Lemma 1, we have $i_n = j_n$. Since $n \geq 1$ was arbitrary, it follows that the two sequences are identical. This shows that the sequence $(i_n)_{n \geq 1}$ is uniquely determined.

Conversely, assume that there is a sequence $(i_n)_{n \geq 1}$ such that $\{x\} = \bigcap_{n \geq 1} I_{i_1, \dots, i_n}$. Since $x \in I_{i_1, \dots, i_n} \subset C_n$ for each $n \geq 1$, it follows that $x \in C$, thereby completing the proof.

(3) Since I_{i_1, \dots, i_n} is closed, each C_n is a finite union of closed sets, so that each C_n is closed. So C is an intersection of closed sets, and it follows that C is closed. Now assume that $x \in C$. Then there is a sequence $(i_n)_{n \geq 1}$ such that $\{x\} = \bigcap_{n \geq 1} I_{i_1, \dots, i_n}$. Let $\epsilon > 0$ be given. Then there exists N such that $e_N < \epsilon$. Define a sequence $(j_n)_{n \geq 1}$ as follows:

$$j_n = \begin{cases} 1 - i_n & \text{if } n = N + 1; \\ i_n & \text{if } n \neq N + 1. \end{cases}$$

Let y be such that $\{y\} = \bigcap_{n \geq 1} I_{j_1, \dots, j_n}$. Then $y \in C$, and by part (2), $y \neq x$. Observe that $x, y \in I_{i_1, \dots, i_N}$; hence $|x - y| \leq d_{i_1, \dots, i_N} - c_{i_1, \dots, i_N} < e_N < \epsilon$, so that $y \in (x - \epsilon, x + \epsilon)$. This shows that x is a limit point of C . Since x was arbitrary, it follows that C is a perfect set. \square

4. THE CANTOR GAME AND THE ZERMELO-FRAENKEL AXIOMS

Using our precise definitions of strategies for players A and B and those properties of generalized Cantor sets given by Theorem 1, we can now prove some remarkably powerful results about limit sets for both players. Once again we note that we are assuming only the axioms of ZF.

First we will show that the limit set of any strategy for player A contains a generalized Cantor set. Then we will use this result to characterize those target sets for which player A has a winning strategy. Next we will show that the limit set of any strategy for player B also contains a generalized Cantor set, and we will use this result to provide necessary conditions for player B to have a winning strategy. Finally, we will introduce the Perfect Set Property and show that it has a close relationship to the Cantor game.

4.1. Results for player A.

Theorem 2. *In the Cantor game on $[a_0, b_0]$ with an uncountable target set S , the limit set of any strategy for player A contains a generalized Cantor set.*

Proof. Suppose that the uncountable target set S for the Cantor game on $[a_0, b_0]$ is given, and let $(f_n)_{n \geq 0}$ be a strategy for player A. Assume that we have constructed a fixed enumeration \mathbb{Q}_0 of the rational numbers in $[a_0, b_0]$. Let $e_n = \frac{b_0 - a_0}{2^n}$ for $n \geq 1$, and define $a_1 = f_0(a_0, b_0)$, $c = a_1$, $d = b_0$, $u = \frac{c+d}{2}$, and $I = [c, d]$. This defines the interval and the bounding sequence that we will use for the construction of a generalized Cantor set.

Let d_1 be the first element of \mathbb{Q}_0 (the one of least index) such that $c < d_1 < u$, and let

$$c_1 = f_1(a_0, b_0, c, d_1).$$

Define d_0 to be the first element of \mathbb{Q}_0 such that $c < d_0 < c_1$, and let

$$c_0 = f_1(a_0, b_0, c, d_0).$$

Also define

$$I_0 = [c_0, d_0];$$

$$I_1 = [c_1, d_1];$$

$$u_0 = \frac{c_0 + d_0}{2};$$

and

$$u_1 = \frac{c_1 + d_1}{2}.$$

Then we have

$$c < c_0 < d_0 < c_1 < d_1 < u < d.$$

Hence

$$I_0, I_1 \subset [c, u] \subset I$$

and

$$I_0 \cap I_1 = \emptyset.$$

By the preceding inequality we have

$$0 < d_{i_1} - c_{i_1} < u - c = \frac{d - c}{2} = \frac{b_0 - a_1}{2} < \frac{b_0 - a_0}{2} = e_1$$

for $i_1 \in \{0, 1\}$. Observe that the left endpoints of I_0 and I_1 are defined by using player A's strategy based upon two rational numbers that might be chosen by player B.

Next we define

$$d_{1,1} = \text{first element of } \mathbb{Q}_0 \text{ such that } c_1 < d_{1,1} < u_1;$$

$$c_{1,1} = f_2(a_0, b_0, c, d_1, c_1, d_{1,1});$$

$$d_{1,0} = \text{first element of } \mathbb{Q}_0 \text{ such that } c_1 < d_{1,0} < c_{1,1};$$

$$c_{1,0} = f_2(a_0, b_0, c, d_1, c_1, d_{1,0});$$

$$d_{0,1} = \text{first element of } \mathbb{Q}_0 \text{ such that } c_0 < d_{0,1} < u_0;$$

$$c_{0,1} = f_2(a_0, b_0, c, d_0, c_0, d_{0,1});$$

$$d_{0,0} = \text{first element of } \mathbb{Q}_0 \text{ such that } c_0 < d_{0,0} < c_{0,1};$$

and

$$c_{0,0} = f_2(a_0, b_0, c, d_0, c_0, d_{0,0}).$$

Let

$$I_{i_1, i_2} = [c_{i_1, i_2}, d_{i_1, i_2}]$$

and

$$u_{i_1, i_2} = \frac{c_{i_1, i_2} + d_{i_1, i_2}}{2}$$

for $i_1, i_2 \in \{0, 1\}$. Observe once again that the left endpoint of each interval I_{i_1, i_2} is defined by using player A's strategy based upon a rational number that might be chosen by player B.

Now assume that for each k with $2 \leq k \leq n$, the collections of real numbers $\{c_{i_1, \dots, i_k} \mid i_1, \dots, i_k \in \{0, 1\}\}$, $\{d_{i_1, \dots, i_k} \mid i_1, \dots, i_k \in \{0, 1\}\}$, and $\{u_{i_1, \dots, i_k} \mid i_1, \dots, i_k \in \{0, 1\}\}$ and the collection of closed intervals $\{I_{i_1, \dots, i_k} \mid i_1, \dots, i_k \in \{0, 1\}\}$ have been defined and satisfy the following properties for all choices of $i_1, i_2, \dots, i_k \in [0, 1]$:

- (1) $d_{i_1, \dots, i_{k-1}, 1} = \text{first element of } \mathbb{Q}_0 \text{ belonging to } (c_{i_1, \dots, i_{k-1}}, u_{i_1, \dots, i_{k-1}});$
- (2) $c_{i_1, \dots, i_{k-1}, 1} = f_k(a_0, b_0, c, d_{i_1}, c_{i_1}, \dots, d_{i_1, \dots, i_{k-1}}, c_{i_1, \dots, i_{k-1}}, d_{i_1, \dots, i_{k-1}, 1});$
- (3) $d_{i_1, \dots, i_{k-1}, 0} = \text{first element of } \mathbb{Q}_0 \text{ belonging to } (c_{i_1, \dots, i_{k-1}}, c_{i_1, \dots, i_{k-1}, 1});$
- (4) $c_{i_1, \dots, i_{k-1}, 0} = f_k(a_0, b_0, c, d_{i_1}, c_{i_1}, \dots, d_{i_1, \dots, i_{k-1}}, c_{i_1, \dots, i_{k-1}}, d_{i_1, \dots, i_{k-1}, 0});$
- (5) $I_{i_1, \dots, i_k} = [c_{i_1, \dots, i_k}, d_{i_1, \dots, i_k}];$ and
- (6) $u_{i_1, \dots, i_k} = \frac{c_{i_1, \dots, i_k} + d_{i_1, \dots, i_k}}{2}.$

Then define collections of real numbers $\{c_{i_1, \dots, i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{0, 1\}\}$, $\{d_{i_1, \dots, i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{0, 1\}\}$, and $\{u_{i_1, \dots, i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{0, 1\}\}$, as well as a collection of closed intervals $\{I_{i_1, \dots, i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{0, 1\}\}$ with the following properties for each choice of $i_1, i_2, \dots, i_{n+1} \in [0, 1]$:

- (1) $d_{i_1, \dots, i_n, 1} = \text{first element of } \mathbb{Q}_0 \text{ belonging to } (c_{i_1, \dots, i_n}, u_{i_1, \dots, i_n});$
- (2) $c_{i_1, \dots, i_n, 1} = f_{n+1}(a_0, b_0, c, d_{i_1}, c_{i_1}, \dots, d_{i_1, \dots, i_n}, c_{i_1, \dots, i_n}, d_{i_1, \dots, i_n, 1});$
- (3) $d_{i_1, \dots, i_n, 0} = \text{first element of } \mathbb{Q}_0 \text{ belonging to } (c_{i_1, \dots, i_n}, c_{i_1, \dots, i_n, 1});$

- (4) $c_{i_1, \dots, i_n, 0} = f_{n+1}(a_0, b_0, c, d_{i_1}, c_{i_1}, \dots, d_{i_1, \dots, i_n}, c_{i_1, \dots, i_n}, d_{i_1, \dots, i_n}, 0)$;
- (5) $I_{i_1, \dots, i_{n+1}} = [c_{i_1, \dots, i_{n+1}}, d_{i_1, \dots, i_{n+1}}]$; and
- (6) $u_{i_1, \dots, i_{n+1}} = \frac{c_{i_1, \dots, i_{n+1}} + d_{i_1, \dots, i_{n+1}}}{2}$.

By induction the first set of six properties holds for the collections of numbers and intervals defined for every $k \geq 2$ and every choice of $i_1, i_2, \dots, i_k \in [0, 1]$.

Now fix $n \geq 2$, and let $k = n$. Applying properties (1) - (4) together with the inequality for f_n in Definition 2, we see that for all choices of $i_1, i_2, \dots, i_{n-1} \in [0, 1]$,

$$c_{i_1, \dots, i_{n-1}} < c_{i_1, \dots, i_{n-1}, 0} < d_{i_1, \dots, i_{n-1}, 0} < c_{i_1, \dots, i_{n-1}, 1}$$

and

$$c_{i_1, \dots, i_{n-1}, 1} < d_{i_1, \dots, i_{n-1}, 1} < u_{i_1, \dots, i_{n-1}} < d_{i_1, \dots, i_{n-1}}.$$

Hence

$$I_{i_1, \dots, i_{n-1}, 0}, I_{i_1, \dots, i_{n-1}, 1} \subset [c_{i_1, \dots, i_{n-1}}, u_{i_1, \dots, i_{n-1}}] \subset I_{i_1, \dots, i_{n-1}}$$

and

$$I_{i_1, \dots, i_{n-1}, 0} \cap I_{i_1, \dots, i_{n-1}, 1} = \emptyset.$$

Combining these relations with those of the construction for $n = 1$, we see that conditions (2) and (3) of Definition 4 are satisfied.

To see that condition (1) of that definition also holds, we proceed by induction. Recall that we have defined $e_n = \frac{b_0 - a_0}{2^n}$ for $n \geq 1$, and we have already shown that $0 < d_{i_1} - c_{i_1} < e_1$ for $i_1 \in \{0, 1\}$. Now assume that $0 < d_{i_1, \dots, i_{n-1}} - c_{i_1, \dots, i_{n-1}} < e_{n-1}$ for all choices of $i_1, \dots, i_{n-1} \in \{0, 1\}$, and let $i_1, \dots, i_n \in \{0, 1\}$ be given. Then $d_{i_1, \dots, i_n} - c_{i_1, \dots, i_n} > 0$ and

$$d_{i_1, \dots, i_n} - c_{i_1, \dots, i_n} < u_{i_1, \dots, i_{n-1}} - c_{i_1, \dots, i_{n-1}} = \frac{d_{i_1, \dots, i_{n-1}} - c_{i_1, \dots, i_{n-1}}}{2} < \frac{e_{n-1}}{2} = e_n.$$

Hence condition (1) of Definition 4 holds.

Now let C be the generalized Cantor set generated by $\mathcal{C} = \{I_{i_1, \dots, i_n} \mid n \geq 1 \text{ and } i_1, \dots, i_n \in \{0, 1\}\}$. We claim that $C \subset L((f_n)_{n \geq 0})$. To prove this, let $x \in C$ be given. By Theorem 1 there is a unique sequence $(i_n)_{n \geq 1}$ such that $\{x\} = \bigcap_{n \geq 1} I_{i_1, \dots, i_n}$. We define a play of the Cantor game as follows. First note that a_0 and b_0 are already given. Now let $a_1 = c$, $b_1 = d_{i_1}$, and define $a_n = c_{i_1, \dots, i_{n-1}}$ and $b_n = d_{i_1, \dots, i_n}$ for all $n \geq 2$. Then $a_1 = f_0(a_0, b_0)$, $a_2 = c_{i_1} = f_1(a_0, b_0, c, d_{i_1}) = f_1(a_0, b_0, a_1, b_1)$, and for $n \geq 3$, $a_n = c_{i_1, \dots, i_{n-1}} = f_{n-1}(a_0, b_0, c, d_{i_1}, c_{i_1}, \dots, d_{i_1, \dots, i_{n-2}}, c_{i_1, \dots, i_{n-2}}, d_{i_1, \dots, i_{n-1}}) = f_{n-1}(a_0, b_0, a_1, b_1, a_2, \dots, b_{n-2}, a_{n-1}, b_{n-1}) = f_{n-1}(a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1})$. Thus the play of the game $((a_n)_{n \geq 0}, (b_n)_{n \geq 0})$ is consistent with the given strategy $(f_n)_{n \geq 0}$ for player A. Now observe that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_{i_1, \dots, i_{n-1}} = x$, so that $x \in L((f_n)_{n \geq 0})$. It follows that the generalized Cantor set we have defined is a subset of player A's limit set. \square

As a consequence of Theorem 2 we obtain a characterization of those target sets S for which player A has a winning strategy.

Theorem 3. *In the Cantor game on $[a_0, b_0]$ with an uncountable target set S , the following three statements are equivalent:*

- (1) *Player A has a winning strategy;*
- (2) *S contains a generalized Cantor set;*
- (3) *S contains a perfect set.*

Proof. (1) \Rightarrow (2) If player A has a winning strategy, then S contains the limit set of that strategy. By Theorem 2, this limit set contains a generalized Cantor set. Thus S contains a generalized Cantor set.

(2) \Rightarrow (3) This is true according to Theorem 1 since every generalized Cantor set is a perfect set.

(3) \Rightarrow (1) This was proved in [2, p.379]. \square

4.2. Results for player B. We now prove that similar results hold for player B's strategy, although the implications of these results will be less far-reaching than the results for player A. The proof of our next result is very similar to the proof of Theorem 2 for player A. The main difference is that the current argument's construction proceeds upward from the midpoint of each subinterval, while the construction in Theorem 2 proceeds downward.

Theorem 4. *In the Cantor game on $[a_0, b_0]$ with an uncountable target set S , the limit set of any strategy for player B contains a generalized Cantor set.*

Proof. Suppose that the uncountable target set S for the Cantor game on $[a_0, b_0]$ is given, and let $(g_n)_{n \geq 0}$ be a strategy for player B. Assume that we have constructed a fixed enumeration \mathbb{Q}_0 of the rational numbers in $[a_0, b_0]$. Let $e_n = \frac{b_0 - a_0}{2^n}$ for $n \geq 1$, and define $c = a_0$, $d = b_0$, $u = \frac{c+d}{2}$, and $I = [c, d]$. This defines the interval and the bounding sequence that we will use for the construction of a generalized Cantor set.

Let c_0 be the first element of \mathbb{Q}_0 (the one of least index) such that $u < c_0 < d$, and let

$$d_0 = g_0(a_0, b_0, c_0).$$

Define c_1 to be the first element of \mathbb{Q}_0 such that $d_0 < c_1 < d$, and let

$$d_1 = g_0(a_0, b_0, c_1).$$

Also define

$$I_0 = [c_0, d_0];$$

$$I_1 = [c_1, d_1];$$

$$u_0 = \frac{c_0 + d_0}{2};$$

and

$$u_1 = \frac{c_1 + d_1}{2}.$$

Then we have

$$c < u < c_0 < d_0 < c_1 < d_1 < d.$$

Hence

$$I_0, I_1 \subset [u, d] \subset I$$

and

$$I_0 \cap I_1 = \emptyset.$$

By the preceding inequality we have

$$0 < d_{i_1} - c_{i_1} < d - u = \frac{d - c}{2} = \frac{b_0 - a_0}{2} = e_1$$

for $i_1 \in \{0, 1\}$. Observe that the right endpoints of I_0 and I_1 are defined by using player B's strategy based upon two rational numbers that might be chosen by player A.

Next we define

$$c_{0,0} = \text{first element of } \mathbb{Q}_0 \text{ such that } u_0 < c_{0,0} < d_0;$$

$$d_{0,0} = g_1(a_0, b_0, c_0, d_0, c_{0,0});$$

$$c_{0,1} = \text{first element of } \mathbb{Q}_0 \text{ such that } d_{0,0} < c_{0,1} < d_0;$$

$$d_{0,1} = g_1(a_0, b_0, c_0, d_0, c_{0,1});$$

$$c_{1,0} = \text{first element of } \mathbb{Q}_0 \text{ such that } u_1 < c_{1,0} < d_1;$$

$$d_{1,0} = g_1(a_0, b_0, c_1, d_1, c_{1,0});$$

$$c_{1,1} = \text{first element of } \mathbb{Q}_0 \text{ such that } d_{1,0} < c_{1,1} < d_1;$$

and

$$d_{1,1} = g_1(a_0, b_0, c_1, d_1, c_{1,1}).$$

Let

$$I_{i_1, i_2} = [c_{i_1, i_2}, d_{i_1, i_2}]$$

and

$$u_{i_1, i_2} = \frac{c_{i_1, i_2} + d_{i_1, i_2}}{2}$$

for $i_1, i_2 \in \{0, 1\}$. Observe once again that the right endpoint of each interval I_{i_1, i_2} is defined by using player B's strategy based upon a rational number that might be chosen by player A.

Now assume that for each k with $2 \leq k \leq n$, the collections of real numbers $\{c_{i_1, \dots, i_k} \mid i_1, \dots, i_k \in \{0, 1\}\}$, $\{d_{i_1, \dots, i_k} \mid i_1, \dots, i_k \in \{0, 1\}\}$, and $\{u_{i_1, \dots, i_k} \mid i_1, \dots, i_k \in \{0, 1\}\}$ and the collection of closed intervals $\{I_{i_1, \dots, i_k} \mid i_1, \dots, i_k \in \{0, 1\}\}$ have been defined and satisfy the following properties for all choices of $i_1, i_2, \dots, i_k \in [0, 1]$:

- (1) $c_{i_1, \dots, i_{k-1}, 0} = \text{first element of } \mathbb{Q}_0 \text{ belonging to } (u_{i_1, \dots, i_{k-1}}, d_{i_1, \dots, i_{k-1}});$
- (2) $d_{i_1, \dots, i_{k-1}, 0} = g_{k-1}(a_0, b_0, c_{i_1}, d_{i_1}, \dots, c_{i_1, \dots, i_{k-1}}, d_{i_1, \dots, i_{k-1}}, c_{i_1, \dots, i_{k-1}, 0});$
- (3) $c_{i_1, \dots, i_{k-1}, 1} = \text{first element of } \mathbb{Q}_0 \text{ belonging to } (d_{i_1, \dots, i_{k-1}, 0}, d_{i_1, \dots, i_{k-1}});$
- (4) $d_{i_1, \dots, i_{k-1}, 1} = g_{k-1}(a_0, b_0, c_{i_1}, d_{i_1}, \dots, c_{i_1, \dots, i_{k-1}}, d_{i_1, \dots, i_{k-1}}, c_{i_1, \dots, i_{k-1}, 1});$
- (5) $I_{i_1, \dots, i_k} = [c_{i_1, \dots, i_k}, d_{i_1, \dots, i_k}];$ and
- (6) $u_{i_1, \dots, i_k} = \frac{c_{i_1, \dots, i_k} + d_{i_1, \dots, i_k}}{2}.$

Then define collections of real numbers $\{c_{i_1, \dots, i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{0, 1\}\}$, $\{d_{i_1, \dots, i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{0, 1\}\}$, and $\{u_{i_1, \dots, i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{0, 1\}\}$, as well as a collection of closed intervals $\{I_{i_1, \dots, i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{0, 1\}\}$ with the following properties for each choice of $i_1, i_2, \dots, i_{n+1} \in [0, 1]$:

- (1) $c_{i_1, \dots, i_n, 0}$ = first element of \mathbb{Q}_0 belonging to $(u_{i_1, \dots, i_n}, d_{i_1, \dots, i_n})$;
- (2) $d_{i_1, \dots, i_n, 0} = g_n(a_0, b_0, c_{i_1}, d_{i_1}, \dots, c_{i_1, \dots, i_n}, d_{i_1, \dots, i_n}, c_{i_1, \dots, i_n, 0})$;
- (3) $c_{i_1, \dots, i_n, 1}$ = first element of \mathbb{Q}_0 belonging to $(d_{i_1, \dots, i_n, 0}, d_{i_1, \dots, i_n})$;
- (4) $d_{i_1, \dots, i_n, 1} = g_n(a_0, b_0, c_{i_1}, d_{i_1}, \dots, c_{i_1, \dots, i_n}, d_{i_1, \dots, i_n}, c_{i_1, \dots, i_n, 1})$;
- (5) $I_{i_1, \dots, i_{n+1}} = [c_{i_1, \dots, i_{n+1}}, d_{i_1, \dots, i_{n+1}}]$; and
- (6) $u_{i_1, \dots, i_{n+1}} = \frac{c_{i_1, \dots, i_{n+1}} + d_{i_1, \dots, i_{n+1}}}{2}$.

By induction the first set of six properties holds for the collections of numbers and intervals defined for every $k \geq 2$ and every choice of $i_1, i_2, \dots, i_k \in [0, 1]$.

Now fix $n \geq 2$, and let $k = n$. Applying properties (1) - (4) together with the inequality for g_n in Definition 2, we see that for all choices of $i_1, i_2, \dots, i_{n-1} \in [0, 1]$,

$$c_{i_1, \dots, i_{n-1}} < u_{i_1, \dots, i_{n-1}} < c_{i_1, \dots, i_{n-1}, 0} < d_{i_1, \dots, i_{n-1}, 0}$$

and

$$d_{i_1, \dots, i_{n-1}, 0} < c_{i_1, \dots, i_{n-1}, 1} < d_{i_1, \dots, i_{n-1}, 1} < d_{i_1, \dots, i_{n-1}}.$$

Hence

$$I_{i_1, \dots, i_{n-1}, 0}, I_{i_1, \dots, i_{n-1}, 1} \subset [u_{i_1, \dots, i_{n-1}}, d_{i_1, \dots, i_{n-1}}] \subset I_{i_1, \dots, i_{n-1}}$$

and

$$I_{i_1, \dots, i_{n-1}, 0} \cap I_{i_1, \dots, i_{n-1}, 1} = \emptyset.$$

Combining these relations with those of the construction for $n = 1$, we see that conditions (2) and (3) of Definition 4 are satisfied.

To see that condition (1) of that definition also holds, we proceed by induction. Recall that we have defined $e_n = \frac{b_0 - a_0}{2^n}$ for $n \geq 1$, and we have already shown that $0 < d_{i_1} - c_{i_1} < e_1$ for $i_1 \in \{0, 1\}$. Now assume that $0 < d_{i_1, \dots, i_{n-1}} - c_{i_1, \dots, i_{n-1}} < e_{n-1}$ for all choices of $i_1, \dots, i_{n-1} \in \{0, 1\}$, and let $i_1, \dots, i_n \in \{0, 1\}$ be given. Then $d_{i_1, \dots, i_n} - c_{i_1, \dots, i_n} > 0$ and

$$d_{i_1, \dots, i_n} - c_{i_1, \dots, i_n} < d_{i_1, \dots, i_{n-1}} - u_{i_1, \dots, i_{n-1}} = \frac{d_{i_1, \dots, i_{n-1}} - c_{i_1, \dots, i_{n-1}}}{2} < \frac{e_{n-1}}{2} = e_n.$$

Hence condition (1) of Definition 4 holds.

Now let D be the generalized Cantor set generated by $\mathcal{D} = \{I_{i_1, \dots, i_n} \mid n \geq 1 \text{ and } i_1, \dots, i_n \in \{0, 1\}\}$. We claim that $D \subset L((g_n)_{n \geq 0})$. To prove this, let $x \in D$ be given. By Theorem 1 there is a unique sequence $(i_n)_{n \geq 1}$ such that $\{x\} = \bigcap_{n \geq 1} I_{i_1, \dots, i_n}$. We define a play of the Cantor game as follows. First note that a_0 and b_0 are fixed. Now let $a_n = c_{i_1, \dots, i_n}$ and $b_n = d_{i_1, \dots, i_n}$ for all $n \geq 1$. Then $b_1 = d_{i_1} = g_0(c, d, c_{i_1}) = g_0(a_0, b_0, a_1)$, and for $n \geq 2$, $b_n = d_{i_1, \dots, i_n} = g_{n-1}(c, d, c_{i_1}, d_{i_1}, \dots, c_{i_1, \dots, i_{n-1}}, d_{i_1, \dots, i_{n-1}}, c_{i_1, \dots, i_n}) = g_{n-1}(a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n)$. Thus the play of the game $((a_n)_{n \geq 0}, (b_n)_{n \geq 0})$ is consistent with the given strategy $(g_n)_{n \geq 0}$ for player B. Because

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_{i_1, \dots, i_n} = x$, it follows that $x \in L((g_n)_{n \geq 0})$. Thus player B's limit set contains the generalized Cantor set D . \square

The following lemma will be useful in several contexts.

Lemma 2. *Let S be an arbitrary subset of the closed interval $[a, b]$, where $a < b$. If S does not contain a perfect set, then $[a, b] - S$ is dense in $[a, b]$.*

Proof. Suppose that S does not contain a perfect set. Let $x \in [a, b]$ be given, and suppose, to reach a contradiction, that $x \notin \overline{[a, b] - S}$. Assume first that $a < x < b$. Then there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset (a, b)$ and $(x - \epsilon, x + \epsilon) \cap ([a, b] - S) = \emptyset$. Hence $(x - \epsilon, x + \epsilon) \subset S$. This implies that S contains the perfect set $[x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]$, contrary to our assumption. Now assume that $x = a$. Then there exists $\epsilon > 0$ such that $[a, a + \epsilon) \cap ([a, b] - S) = \emptyset$. This implies that S contains the perfect set $[a + \frac{\epsilon}{4}, a + \frac{3\epsilon}{4}]$, contrary to our assumption. The case in which $x = b$ is handled similarly. It follows that $x \in \overline{[a, b] - S}$. This proves that $\overline{[a, b] - S} = [a, b]$. \square

As a consequence of Theorem 4 and Lemma 2 we obtain necessary conditions for player B to have a winning strategy.

Theorem 5. *In the Cantor game on $[a_0, b_0]$ with an uncountable target set S , if player B has a winning strategy, then:*

- (1) $[a_0, b_0] - S$ contains a generalized Cantor set;
- (2) $[a_0, b_0] - S$ contains a perfect set;
- (3) $[a_0, b_0] - S$ is dense in $[a_0, b_0]$.

Proof. (1) If player B has a winning strategy, then the complement of S contains the limit set of that strategy. By Theorem 4, this limit set contains a generalized Cantor set. Thus the complement of S contains a generalized Cantor set.

(2) This is true according to Theorem 1 since every generalized Cantor set is a perfect set.

(3) Since player B has a winning strategy, player A cannot have a winning strategy. Then by Theorem 3, S does not contain a perfect set. It now follows from Lemma 2 that $[a_0, b_0] - S$ is dense in $[a_0, b_0]$. \square

4.3. The Perfect Set Property. We now introduce the Perfect Set Property and show that it has a close relationship to the Cantor game.

Definition 5. *Let $S \subset \mathbb{R}$. We say that S has the perfect set property if and only if S is countable or there exists a nonempty perfect set $P \subset S$. Equivalently, if S is uncountable, then S must contain a nonempty perfect set.*

Up to this point we have been assuming only the eight axioms of Zermelo-Fraenkel set theory (ZF); however, in our next theorem we will need to know that a countable union of countable sets is countable. Since this result is not known to be provable in ZF, we will need to assume an additional axiom

of set theory in order to use it. One such axiom is the Axiom of Countable Choice, AC_ω , which is called the Countable Axiom of Choice by some authors. A precise statement of this axiom and some of its consequences can be found in Jech's monograph [4, Chapter 5].

Theorem 6. *Assuming the Axiom of Countable Choice in addition to ZF ($ZF + AC_\omega$), the following two statements are equivalent:*

- (1) *Every subset of \mathbb{R} has the perfect set property.*
- (2) *For every $a_0, b_0 \in \mathbb{R}$ with $a_0 < b_0$ and for every uncountable set $S \subset [a_0, b_0]$, player A has a winning strategy in the Cantor game on $[a_0, b_0]$ with target set S . (That is, player A always has a winning strategy.)*

Proof. (1) \Rightarrow (2) Suppose that every subset of \mathbb{R} has the perfect set property. Let $a_0, b_0 \in \mathbb{R}$ with $a_0 < b_0$, and let S be any uncountable subset of $[a_0, b_0]$. Then S contains a nonempty perfect set, and so by Theorem 3, player A has a winning strategy in the Cantor game on $[a_0, b_0]$ with target set S .

(2) \Rightarrow (1) Now assume that condition (2) holds. Let $S \subset \mathbb{R}$, and assume that S is uncountable. If $S \cap [n, n+1]$ were countable for every $n \in \mathbb{Z}$, then $S = \bigcup_{n \in \mathbb{Z}} S \cap [n, n+1]$ would be countable. Hence $S \cap [n, n+1]$ must be uncountable for some $n \in \mathbb{Z}$. By condition (2), in the Cantor game on $[n, n+1]$ with target set $S \cap [n, n+1]$, player A has a winning strategy. Hence, by Theorem 3, $S \cap [n, n+1]$ contains a nonempty perfect set, so that S also contains a perfect set. Thus every subset of \mathbb{R} has the perfect set property. \square

5. THE CANTOR GAME AND THE AXIOM OF DETERMINACY

Prior to Section 4.3 we assumed only the eight axioms of Zermelo-Fraenkel set theory (ZF), and in that section we added the assumption of the Axiom of Countable Choice (AC_ω). In order to provide definitive answers to Baker's three questions, we must assume axioms in addition to those of $ZF + AC_\omega$. As we will see, the answers to these questions depend on which additional axioms we assume.

In this brief section we will assume that the Axiom of Determinacy (AD) holds in addition to the Zermelo-Fraenkel axioms, and thus the results will be valid in the $ZF + AD$ system. In the following section we will assume that the Axiom of Choice holds in addition to the Zermelo-Fraenkel axioms, and the results in that section will be valid in the ZFC system. Since the Axiom of Countable Choice (AC_ω) is a consequence of the Axiom of Determinacy and of the Axiom of Choice, all of our previous results will continue to hold in this section and the following one. In Chapter 5 and Chapter 33 of his treatise [4], Jech offers precise statements of all of these axioms and many of their consequences. The most significant consequence for our purposes is the following.

Theorem 7. *Assuming the Axiom of Determinacy in addition to ZF (ZF + AD), every subset of \mathbb{R} has the perfect set property.*

Combining this result with Theorem 6, we obtain the following result.

Theorem 8. *Assuming the Axiom of Determinacy in addition to ZF (ZF + AD), for every $a_0, b_0 \in \mathbb{R}$ with $a_0 < b_0$ and for every uncountable set $S \subset [a_0, b_0]$, player A has a winning strategy in the Cantor game on $[a_0, b_0]$ with target set S .*

Proof. By Theorem 7, (1) of Theorem 6 holds, and so (2) of that theorem holds. \square

As a corollary to Theorem 8, we find that in ZF + AD, the answer to all three questions is “no.”

Corollary 1. *Assuming the Axiom of Determinacy in addition to the Zermelo-Fraenkel axioms (ZF + AD), let $a_0, b_0 \in \mathbb{R}$ with $a_0 < b_0$, and consider the Cantor game on $[a_0, b_0]$. Then*

- (1) *There does not exist an uncountable set $S \subset [a_0, b_0]$ such that player A does not have a winning strategy;*
- (2) *There does not exist an uncountable set $S \subset [a_0, b_0]$ such that player B has a winning strategy;*
- (3) *There does not exist an uncountable set $S \subset [a_0, b_0]$ such that neither player A nor player B has a winning strategy.*

Proof. All three of these assertions follow from the conclusion of Theorem 8 that player A has a winning strategy for every uncountable set $S \subset [a_0, b_0]$. \square

The Axiom of Determinacy includes the definition of a set being determined. Since the Axiom of Determinacy applies to infinite games defined in terms of collections of sequences of natural numbers, and since the Cantor game involves increasing sequences of real numbers and their limits, that definition is not directly applicable. Nevertheless, it is instructive to formulate an analogous definition for the Cantor game.

Definition 6. *Let $S \subset [a_0, b_0]$ with $a_0 < b_0$. We say that the set S is determined if either player A or player B has a winning strategy in the Cantor game with target set S . Otherwise, the set S is said to be non-determined.*

According to Theorem 8 every uncountable subset of $[a_0, b_0]$ is determined. Since player B has a winning strategy for every countable subset of $[a_0, b_0]$ (see [2, p.377]), it follows that every subset of $[a_0, b_0]$ is determined when we assume the axioms of ZF + AD. Thus under the assumption of the Axiom of Determinacy, our results offer a parallel to the statement of that axiom.

6. THE CANTOR GAME AND THE AXIOM OF CHOICE

If we assume the Axiom of Choice instead of the Axiom of Determinacy in addition to the Zermelo-Fraenkel axioms, we obtain an entirely different set of answers to Baker's three questions. Since our answers to two of these questions require the concept of a Bernstein set of real numbers, we begin with some relevant definitions and results. Throughout this section we assume that the Axiom of Choice holds in addition to the Zermelo-Fraenkel axioms. Thus the results will be valid in ZFC.

6.1. Bernstein sets. In Theorem 5.3 [3, p.24] Oxtoby defines the notion of a Bernstein set of real numbers. Here we extend his definition to subsets of \mathbb{R} .

Definition 7. *Let $X \subset \mathbb{R}$, and let X have the subspace topology inherited from \mathbb{R} . A Bernstein set in X is a set $S \subset X$ such that neither S nor $X - S$ contains an uncountable closed set.*

Oxtoby also proves the following result. His proof uses the Well-ordering Principle, which is equivalent to the Axiom of Choice.

Theorem 9. *There exists a set B of real numbers such that B is a Bernstein set in \mathbb{R} .*

Our first lemma provides us with a convenient method of making new Bernstein sets from existing ones.

Lemma 3. *Let X and Y be uncountable closed subsets of \mathbb{R} with $Y \subset X$. If B is a Bernstein set in X , then $B' = B \cap Y$ is a Bernstein set in Y .*

Proof. Since Y is uncountable and closed, $B' \neq \emptyset$. Let C be any uncountable closed set in Y . Write $C = D \cap Y$, where D is an uncountable closed set in X . We see that C is an uncountable closed set in X , so that $B \cap C \neq \emptyset$ and $(X - B) \cap C \neq \emptyset$. Observe that if $x \in B \cap C$, then $x \in Y$, and so $x \in B' \cap C$. Now if $y \in (X - B) \cap C$, then $y \in Y$. Since $B' \subset B$ and $y \notin B$, we have $y \notin B'$. Hence $y \in (Y - B') \cap C$. This shows that B' is a Bernstein set in Y . \square

Using these results we prove the existence of a Bernstein set in a closed interval.

Corollary 2. *There exists a Bernstein set B' in the closed interval $[a, b]$, where $a < b$.*

Proof. Suppose that B is a Bernstein set in \mathbb{R} , and let $B' = B \cap [a, b]$. Applying Lemma 3 with $X = \mathbb{R}$ and $Y = [a, b]$ shows that B' is a Bernstein set in $[a, b]$. \square

We now show that intersecting a Bernstein set in $[a, b]$ with a subinterval of $[a, b]$ yields a Bernstein set in that subinterval. We shall need this property in a later example.

Corollary 3. *Let B' be a Bernstein set in the closed interval $[a, b]$, where $a < b$, and let $[c, d]$ be any closed subinterval of $[a, b]$ with $a \leq c < d \leq b$. Then $B'' = B' \cap [c, d]$ is a Bernstein set in $[c, d]$.*

Proof. This is a special case of Lemma 3 with $X = [a, b]$ and $Y = [c, d]$. \square

Bernstein sets have many interesting and pathological properties. One property that will be essential later is the following.

Theorem 10. *Let B' be a Bernstein set in the closed interval $[a, b]$, where $a < b$. Then B' is non-Lebesgue measurable and hence uncountable.*

Proof. Our argument is similar to that given by Oxtoby [3, p.24] for his Bernstein set $B \subset \mathbb{R}$. Suppose, to arrive at a contradiction, that B' is Lebesgue measurable, and let K be any compact subset of B' . Since K is closed, K must be countable, and so the Lebesgue measure $m(K) = 0$. It follows that $m(B') = \sup\{m(K) \mid K \subset B', K \text{ compact}\} = 0$. Since $[a, b] - B'$ is also Lebesgue measurable, similar reasoning shows that $m([a, b] - B') = 0$. This yields the contradiction that $b - a = m([a, b]) = m(B') + m([a, b] - B') = 0$. Thus B' cannot be Lebesgue measurable. Since every countable set is Lebesgue measurable and has measure 0, B' must also be uncountable. \square

The following theorem will also be helpful.

Theorem 11. *Let B' be a Bernstein set in the closed interval $[a, b]$, where $a < b$. Then B' and $[a, b] - B'$ are dense in $[a, b]$.*

Proof. By definition, B' does not contain an uncountable closed set. Since every perfect set is uncountable and closed, B' does not contain a perfect set. By Lemma 2, $[a, b] - B'$ is dense in $[a, b]$. Similarly, $[a, b] - B'$ does not contain a perfect set, and so by Lemma 2, $[a, b] - ([a, b] - B') = B'$ is dense in $[a, b]$. \square

6.2. Examples. In this section we consider a pair of examples. Our first example uses the following theorem and its corollary, which is sometimes referred to as the Cantor-Bendixson Theorem.

Theorem 12. *Let S be an uncountable subset of \mathbb{R} . Then there is a perfect set P such that $S \cap P$ is uncountable and $S - P$ is countable.*

Proof. This is proved in Exercise 2.27 of [5, p.45]. The proof shows that in fact $P = C(S)$, the set of all condensation points of S . See Definition 8 in Section 6.3 for a definition of $C(S)$ and related sets. \square

Corollary 4. *Let S be a closed and uncountable subset of \mathbb{R} . Then there exists a perfect set P and a countable set D such that $S = P \cup D$.*

Proof. Let $P = C(S)$ be the set defined in Theorem 12, and let $D = S - P$. Then P is a perfect set and D is countable. Since S is closed, $P = C(S) \subset S$, and so $S = P \cup (S - P) = P \cup D$. \square

We are now ready for our first example, which illustrates Theorem 3.

Example 1. Consider the Cantor game on the interval $[0, 1]$ with target set $S = (\mathbb{R} - \mathbb{Q}) \cap [0, 1]$; that is, S is the set of irrational numbers in $[0, 1]$. We show that player A has a winning strategy.

Proof. Since $\mathbb{Q} \cap [0, 1]$ is countable, we may write $\mathbb{Q} \cap [0, 1] = \{q_n \mid n \geq 1\}$. For each $n \geq 1$ let

$$U_n = (q_n - \frac{1}{2^{n+2}}, q_n + \frac{1}{2^{n+2}}) \cap [0, 1]$$

and define

$$U = \bigcup_{n \geq 1} U_n.$$

We see that $m(U) \leq \sum_{n=1}^{\infty} m(U_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$. Now define $C = [0, 1] - U$. Since $\mathbb{Q} \cap [0, 1] \subset U$, $C \subset S$, and we have $m(C) = 1 - m(U) \geq \frac{1}{2}$. Hence C is uncountable. Since U is the union of open sets in $[0, 1]$, U is open, and so C is closed. By Corollary 4 we may write $C = P \cup D$, where P is a perfect set and D is a countable set; then $P \subset C \subset S$. Since S contains the perfect set P , player A has a winning strategy by Theorem 3. \square

Now the following example shows that the converse of Theorem 5 is not true.

Example 2. Let S be a Bernstein set in $[0, \frac{1}{2}]$. Now consider S as a subset of $[0, 1]$. We show that the set $[0, 1] - S$ satisfies the three conditions of Theorem 5 and that player B does not have a winning strategy when S is the target set for the Cantor game on $[0, 1]$.

Proof. We first observe that $[0, 1] - S = (\frac{1}{2}, 1] \cup ([0, \frac{1}{2}] - S)$ and that both S and $[0, \frac{1}{2}] - S$ do not contain perfect sets. Since $(\frac{1}{2}, 1]$ contains the perfect set $[\frac{3}{4}, 1]$, $[0, 1] - S$ contains a perfect set. If we construct a Cantor middle-thirds set, which is a generalized Cantor set, inside $[\frac{3}{4}, 1]$, then $[0, 1] - S$ contains this set. Finally, since S does not contain a perfect set, it follows from Lemma 2 that $[0, 1] - S$ is dense in $[0, 1]$. Thus the three necessary conditions in Theorem 5 are satisfied.

To see that player B does not have a winning strategy, suppose that $(g_n)_{n \geq 0}$ is any strategy for player B. Let $s \in S$ with $s > 0$ be given. First define $a_0 = 0$ and $b_0 = 1$, and note that $b_0 > s > 0$. Next define $a_1 = \frac{s}{2} = (1 - \frac{1}{2^1})s$ and

$$b_1 = \begin{cases} 0 & \text{if } g_0(a_0, b_0, a_1) < s; \\ g_0(a_0, b_0, a_1) & \text{if } g_0(a_0, b_0, a_1) \geq s. \end{cases}$$

Then define $a_2 = \frac{3s}{4} = (1 - \frac{1}{2^2})s$ and

$$b_2 = \begin{cases} 0 & \text{if } b_1 = 0; \\ 0 & \text{if } b_1 > 0 \text{ and } g_1(a_0, b_0, a_1, b_1, a_2) < s; \\ g_1(a_0, b_0, a_1, b_1, a_2) & \text{if } b_1 > 0 \text{ and } g_1(a_0, b_0, a_1, b_1, a_2) \geq s. \end{cases}$$

Now suppose that $n \geq 3$, that $a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1}$ have been defined, and that $a_i = (1 - \frac{1}{2^i})s$ for $1 \leq i \leq n-1$. Define $a_n = (1 - \frac{1}{2^n})s$, $c_n = g_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1}, a_n)$, and

$$b_n = \begin{cases} 0 & \text{if } b_{n-1} = 0; \\ 0 & \text{if } b_{n-1} > 0 \text{ and } c_n < s; \\ c_n & \text{if } b_{n-1} > 0 \text{ and } c_n \geq s. \end{cases}$$

By induction we have defined sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ such that $a_n = (1 - \frac{1}{2^n})s$ for all $n \geq 0$ and either (i) $b_n = g_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1}, a_n) \geq s$ for all $n \geq 1$ or (ii) there exists an integer $N \geq 0$ such that $b_n = g_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1}, a_n) \geq s$ for all $n \leq N$ and $b_n = 0$ for all $n > N$. We now show that player B does not have a winning strategy in either case.

In case (i) we see that $((a_n)_{n \geq 0}, (b_n)_{n \geq 0})$ is a play of the Cantor game on $[0, 1]$ which is consistent with the strategy of player B. Since $\lim_{n \rightarrow \infty} a_n = s \in S$, player A wins this particular game, and so $(g_n)_{n \geq 0}$ is not a winning strategy for player B.

Now consider case (ii). In this case we define $\hat{a}_0 = a_N$, $\hat{b}_0 = b_N$, and $\hat{S} = S \cap [\hat{a}_0, \hat{b}_0]$. Observe that \hat{S} is a Bernstein set by Corollary 3. For the Cantor game on $[\hat{a}_0, \hat{b}_0]$ with target set \hat{S} , define a new strategy for player B $(\hat{g}_n)_{n \geq 0}$ as follows. Let $\hat{g}_0(\hat{a}_0, \hat{b}_0, \bar{a}_1) = g_N(a_0, b_0, a_1, b_1, \dots, a_N, b_N, \bar{a}_1)$, $\hat{g}_1(\hat{a}_0, \hat{b}_0, \bar{a}_1, \bar{b}_1, \bar{a}_2) = g_{N+1}(a_0, b_0, a_1, b_1, \dots, a_N, b_N, \bar{a}_1, \bar{b}_1, \bar{a}_2)$, and for $n \geq 3$, $\hat{g}_{n-1}(\hat{a}_0, \hat{b}_0, \bar{a}_1, \bar{b}_1, \dots, \bar{a}_{n-1}, \bar{b}_{n-1}, \bar{a}_n) = g_{N+n-1}(a_0, b_0, \dots, a_N, b_N, \bar{a}_1, \bar{b}_1, \dots, \bar{a}_{n-1}, \bar{b}_{n-1}, \bar{a}_n)$ for all real numbers $\bar{a}_1, \dots, \bar{a}_n$ and $\bar{b}_1, \dots, \bar{b}_{n-1}$ satisfying $\hat{a}_0 < \bar{a}_1 < \dots < \bar{a}_n < \bar{b}_{n-1} < \bar{b}_{n-2} < \dots < \hat{b}_0$.

Since \hat{S} is a Bernstein set, the strategy $(\hat{g}_n)_{n \geq 0}$ is not a winning strategy for player B. Let $((\hat{a}_n)_{n \geq 0}, (\hat{b}_n)_{n \geq 0})$ be a play of the Cantor game on $[\hat{a}_0, \hat{b}_0]$ consistent with (\hat{g}_n) such that player A wins. Then $\hat{a} = \lim_{n \rightarrow \infty} \hat{a}_n \in \hat{S} \subset S$. Define $a_n = \hat{a}_{n-N}$ and $b_n = \hat{b}_{n-N}$ for all $n > N$. By the way that N was chosen, we have $b_n = g_{n-1}(a_0, b_0, \dots, a_{n-1}, b_{n-1}, a_n)$ for all $n \leq N$. Now if $n > N$, then $b_n = \hat{b}_{n-N} = \hat{g}_{n-N-1}(\hat{a}_0, \hat{b}_0, \hat{a}_1, \hat{b}_1, \dots, \hat{a}_{n-N-1}, \hat{b}_{n-N-1}, \hat{a}_{n-N}) = g_{n-1}(a_0, b_0, \dots, a_N, b_N, \hat{a}_1, \hat{b}_1, \dots, \hat{a}_{n-N-1}, \hat{b}_{n-N-1}, \hat{a}_{n-N}) = g_{n-1}(a_0, b_0, \dots, a_N, b_N, a_{N+1}, b_{N+1}, \dots, a_{n-1}, b_{n-1}, a_n)$. Thus the play of the game $((a_n)_{n \geq 0}, (b_n)_{n \geq 0})$ is consistent with the original strategy $(g_n)_{n \geq 0}$. Since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \hat{a}_n = \hat{a} \in S$, player A wins. It follows that $(g_n)_{n \geq 0}$ is not a winning strategy for player B. \square

6.3. Answers in ZFC. We now show that in ZFC, where the concept of a Bernstein set is defined, the answer to questions 1 and 3 is “yes.”

Corollary 5. *Let S be the target set for the Cantor game on $[a_0, b_0]$. If S is a Bernstein set in $[a_0, b_0]$, then player A does not have a winning strategy.*

Proof. Since every perfect set is both uncountable and closed, if S is a Bernstein set, then S does not contain a perfect set. So by Theorem 3, player A cannot have a winning strategy in this case. \square

Thus Corollary 5 shows that the answer to the first question is “yes,” and indeed Theorem 3 characterizes all of the sets for which player A does not have a winning strategy. Now the following corollary shows that the answer to question 3 is also “yes.”

Corollary 6. *Let S be the target set for the Cantor game on $[a_0, b_0]$. If S is a Bernstein set in $[a_0, b_0]$, then neither player A nor player B has a winning strategy.*

Proof. We already proved in Corollary 5 that player A does not have a winning strategy. Since every perfect set is both uncountable and closed, if S is a Bernstein set, then the complement of S does not contain a perfect set. So by Theorem 5, player B cannot have a winning strategy. \square

Recalling our discussion of determinacy following Corollary 1, Corollary 6 states that every Bernstein set in $[a_0, b_0]$ is non-determined when we assume the axioms of ZFC. Thus, under the assumption of the Axiom of Choice, our results run parallel to results in descriptive set theory in which the Axiom of Choice contradicts the Axiom of Determinacy. Chapter 33 of Jech’s monograph [4] contains an excellent discussion of these results.

Example 2 showed that the necessary conditions in Theorem 5 are not sufficient to ensure that player B has a winning strategy. We now investigate whether or not player B can ever have a winning strategy. To do so we consider the condensation points of the target set S and how these may relate to winning strategies for player B.

Definition 8. *Let $a_0, b_0 \in \mathbb{R}$ with $a_0 < b_0$, and let $S \subset [a_0, b_0]$. Define*

$$C_+(S) = \{x \in [a_0, b_0] \mid (x, x + \epsilon) \cap S \text{ is uncountable for every } \epsilon > 0\}$$

to be the set of all right condensation points of S , and define

$$C_-(S) = \{x \in [a_0, b_0] \mid (x - \epsilon, x) \cap S \text{ is uncountable for every } \epsilon > 0\}$$

to be the set of all left condensation points of S . Then

$$C(S) = C_+(S) \cup C_-(S)$$

is the set of all condensation points of S .

Now define the set of all two-sided condensation points of S as

$$T(S) = C_+(S) \cap C_-(S);$$

the set of left-only condensation points of S as

$$L(S) = C_-(S) - C_+(S);$$

the set of right-only condensation points of S as

$$R(S) = C_+(S) - C_-(S);$$

and the set of one-sided condensation points of S as

$$O(S) = L(S) \cup R(S).$$

The following results will be helpful in connecting winning strategies for player B and the condensation points of the set S .

Theorem 13. *Let $S \subset [a_0, b_0]$, where $a_0 < b_0$. If S is uncountable, then $S - C_+(S)$ is countable, and so $S \cap C_+(S)$ and $C_+(S)$ are uncountable.*

Proof. The proof of Exercise 2.27 in [5, p.45], to which we referred in the proof of Theorem 12, also implies that $L(S)$ and $R(S)$ are countable, so that $O(S) = L(S) \cup R(S)$ is countable. By Theorem 12 $S \cap C(S)$ is uncountable and $S - C(S)$ is countable. It follows that $S - T(S) = S - (C(S) - O(S)) = (S - C(S)) \cup (S \cap O(S))$ is countable. Now we have $T(S) \subset C_+(S)$, so that $S - C_+(S) \subset S - T(S)$ is countable. Since $S = (S \cap C_+(S)) \cup (S - C_+(S))$ and S is uncountable, $S \cap C_+(S)$ is uncountable, so that $C_+(S)$ is also uncountable. \square

Theorem 14. *If $x \in S \cap C_+(S)$ and $x < y$, then there are uncountably many points z in $S \cap C_+(S)$ with $x < z < y$.*

Proof. Suppose that $x \in S \cap C_+(S)$ and $y \in \mathbb{R}$ is such that $x < y$. If $\epsilon = y - x$, then $(x, x + \epsilon) = (x, y)$ contains uncountably many points of S . Since $S - C_+(S)$ is countable by Theorem 13, (x, y) must contain uncountably many points in $S \cap C_+(S)$. \square

Our next result reveals the connection between the countability of the set S and the set of its condensation points.

Theorem 15. *Let $S \subset [a_0, b_0]$, where $a_0 < b_0$. Then S is countable if and only if $C_+(S) = \emptyset$.*

Proof. If S is countable, then clearly $(x, x + \epsilon) \cap S$ cannot be uncountable for any x ; hence $C_+(S) = \emptyset$. Now assume that S is uncountable. By Theorem 13, $C_+(S)$ is uncountable, so that $C_+(S) \neq \emptyset$. \square

Corollary 7. *If $C_+(S) = \emptyset$, then player B has a winning strategy.*

Proof. If $C_+(S) = \emptyset$, then Theorem 15 implies that S is countable. By [2, p.377], player B has a winning strategy. \square

Conjecture. *If $C_+(S) \neq \emptyset$, then player B does not have a winning strategy.*

Theorem 15, Corollary 7, and this conjecture imply that player B has a winning strategy if and only if S is countable. Thus if this conjecture is correct, then the answer to question 2 is “no” when we assume the axioms of ZFC.

To see that this conjecture is very likely to be true, we begin by showing that player A may choose the sequence $(a_n)_{n \geq 0}$ so that for every $n \geq 1$ there are uncountably many choices of a_n from the set $S \cap C_+(S)$.

Let $S \subset [a_0, b_0]$, where $a_0 < b_0$, be an uncountable set, and define $c_0 = \inf(S \cap C_+(S))$. Then $a_0 \leq c_0$, and since $S \cap C_+(S)$ is uncountable according to Theorem 13, $c_0 < b_0$. If $c_0 \in S \cap C_+(S)$, then by Theorem 14 there are uncountably many points $a_1 \in S \cap C_+(S)$ such that $a_0 \leq c_0 < a_1 < b_0$. If

$c_0 \notin S \cap C_+(S)$, then we may choose a point $c_1 \in S \cap C_+(S)$ such that $c_0 < c_1 < b_0$. By Theorem 14 there are uncountably many points $a_1 \in S \cap C_+(S)$ such that $a_0 \leq c_0 < c_1 < a_1 < b_0$. In either case, assume that player A has selected one of these points a_1 .

Now suppose that b_1 has been chosen such that $a_1 < b_1 < b_0$. Again by Theorem 14, since $a_1 \in S \cap C_+(S)$, there are uncountably many points $a_2 \in S \cap C_+(S)$ with $a_1 < a_2 < b_1$. Assume that player A has selected one of these points a_2 and that player B has chosen b_2 such that $a_2 < b_2 < b_1$. Continuing this process we see that for each $n \geq 1$ player A has uncountably many choices for a_n from the set $S \cap C_+(S)$.

In order to show that player B does not have a winning strategy, we need only find a single play of the Cantor game in which player A wins. Recall that the winner of a play of the Cantor game is determined by the value of $a = \lim_{n \rightarrow \infty} a_n$. When $a \in S$, player A wins, and when $a \in [a_0, b_0] - S$, player B wins. When the sequence $(a_n)_{n \geq 0}$ is chosen so that $a_n \in S \cap C_+(S)$ for all $n \geq 1$, $a \in \overline{S \cap C_+(S)} \subset \overline{S \cap C(S)} \subset \overline{C(S)} = C(S)$. Since $S \cap C_+(S)$ is uncountable, $S \cap C(S)$ is also uncountable. Thus there are uncountably many ways that player A may choose each term of the sequence (a_n) and uncountably many potential limits in $S \cap C(S)$ that will yield a play of the game that player A wins. Each time player B chooses a value for the next term b_n , player A's choices are restricted, yet player A maintains having uncountably many choices from $S \cap C_+(S)$ at each step. So it seems very likely that at least one such sequence will result in player A winning a game.

Recall from Theorem 5 that if player B has a winning strategy, then $[a_0, b_0] - S$ is dense in $[a_0, b_0]$ and contains a perfect set. In order for player B to control the outcome of a play of the Cantor game, player B's chosen sequence $(b_n)_{n \geq 0}$ must force player A's sequence $(a_n)_{n \geq 0}$ to converge to a point of $[a_0, b_0] - S$. This seems to require that $b = \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n = a$ and that the points b_n be chosen from a subset of $[a_0, b_0] - S$ that is both closed and dense in some subinterval of $[a_0, b_0]$. Thus $[a_0, b_0] - S$ would have to contain a closed subinterval that includes points of player A's chosen sequences, but this is impossible when player A's sequences are chosen from $S \cap C_+(S)$.

Although it may be possible to employ the Axiom of Choice directly to construct a play of the Cantor game that player A wins, a more promising approach is to use Zorn's Lemma. In this approach we consider increasing sequences of real numbers in $[a_0, b_0]$ which determine valid plays of the Cantor game on $[a_0, b_0]$, and we define a preordering on some collection of such sequences. If we can show that each chain of elements in that collection has an upper bound which also lies in the collection, we can conclude from Zorn's Lemma that this collection possesses a maximal element. If we can then show that this maximal element must have its limit in the set S , then we will have found a play of the Cantor game that player A wins. This will prove that player B never has a winning strategy, and we believe that it will be a "winning strategy" for constructing a proof of our conjecture.

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REFERENCES

- [1] J. W. Grossman and B. Turett, Problem #1542, *Mathematics Magazine*. **71** (1998), 67.
- [2] Matthew H. Baker, *Uncountable Sets and an Infinite Real Number Game*, *Mathematics Magazine*. **80** (2007), 377–380.
- [3] John C. Oxtoby, *Measure and Category*. 2nd ed., Springer-Verlag, New York, 1980.
- [4] Thomas Jech, *Set Theory: The Third Millennium Edition*, revised and expanded, Springer-Verlag, New York, 2003.
- [5] Walter Rudin, *Principles of Mathematical Analysis*. 3rd ed., McGraw-Hill, New York, 1976.